Solutions for Exercise 1

1. Suppose that the balls are labelled S_1, S_2, S_3, G_1, G_2 with an obvious convention for their colours. A suitable sample space is the set of all possible pairs of the 5 balls, with order taken into account. There are 25 such possible pairs. This gives

$$\Omega = \{(S_1, S_1), (S_1, S_2), \ldots, (G_2, G_2)\}$$

They can be put into a table as follows:

	First Ball				
	S_1	S_2	S_3	G_1	G_2
Second Ball					
S_1	○	○	○	○	○
S_2	○	○	○	○	○
S_3	○	○	○	○	○
G_1	○	○	○	○	○
G_2	○	○	○	○	○

Each dot represents an outcome in Ω. We can mark the outcomes in events E_1 with $\square$ and E_2 with $\triangle$.

	First Ball				
	S_1	S_2	S_3	G_1	G_2
Second Ball					
S_1	$\square\triangle$	$\square\triangle$	$\square\triangle$	$\triangle$	$\triangle$
S_2	$\square\triangle$	$\square\triangle$	$\square\triangle$	$\triangle$	$\triangle$
S_3	$\square\triangle$	$\square\triangle$	$\square\triangle$	$\triangle$	$\triangle$
G_1	$\square$	$\square$	$\square$	○	○
G_2	$\square$	$\square$	$\square$	○	○

To find $P(E_1)$, $P(E_2)$ and $P(E_1 \cap E_2)$, since the outcomes in the sample space are equally likely, it is enough to count them.

$$P(E_1) = 15/25 = 3/5.$$

$$P(E_2) = 15/25 = 3/5.$$

$$P(E_1 \cap E_2) = 9/25.$$

Notice that $P(E_1 \cap E_2) = P(E_1)P(E_2)$ so that E_1 and E_2 are *independent* events.

If we sample without replacement, then the outcomes on the diagonal of the table are no longer needed. All the others remain equally likely.

	First Ball				
	S_1	S_2	S_3	G_1	G_2
Second Ball					
S_1		$\square\triangle$	$\square\triangle$	$\triangle$	$\triangle$
S_2	$\square\triangle$		$\square\triangle$	$\triangle$	$\triangle$
S_3	$\square\triangle$	$\square\triangle$		$\triangle$	$\triangle$
G_1	$\square$	$\square$	$\square$		○
G_2	$\square$	$\square$	$\square$	○	

Recounting the outcomes

$$P(E_1) = 12/20 = 3/5.$$

$$P(E_2) = 12/20 = 3/5.$$

$$P(E_1 \cap E_2) = 6/20 = 3/10.$$

This time the events are not independent, but we can clearly see that $P(E_1) = P(E_2)$, which is not otherwise completely transparent for sampling without replacement.

2. The probabilities of not inspecting the two faulty motors are easily seen to be

   (a) $\binom{10}{2}/\binom{12}{2} = (10 \times 9)/(12 \times 11) = 15/22$.

   (b) $(\binom{5}{1}/\binom{6}{1})^2 = 25/36$.

   (c) $(\binom{4}{1}/\binom{6}{1}) \times 1 = 2/3$.

   The second one is the largest.

3. (a) There are 8 equally likely ordered arrangements of heads and tails when 3 coins are tossed. All except for HHH and TTT have an "odd one out". So the probability of having a loser on a given turn is $6/8 = 3/4$.

   (b) The probability of an even number of turns is

$$\sum_{i=1}^{\infty} P(2i \text{ turns needed})$$

   which is

$$\sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^{2i-1} \frac{3}{4} = \frac{3}{16} \sum_{i=0}^{\infty} \left(\frac{1}{16}\right)^{i} = (3/16)/(1 - 1/16) = 1/5.$$

   Notice that we need to add here the probabilities for a whole sequence of disjoint events to get the answer.

4. There are a lot of ways of doing this. A throws $n+1$ times and B throws n times. Consider A's first n throws together with B's throws; a total of $2n$ throws, each having 2 possible outcomes, so there is a total of 2^{2n} equally likely outcomes. Now let H_A be the number of heads in A's first n throws and H_B be the number of heads in B's throws. We can partition the sample space into three events;

   E_1 event that $H_A > H_B$ E_2 event that $H_A = H_B$ E_3 event that $H_A < H_B$

   By mutual exclusive and exhaustive properties, the sum of the number of outcomes in each event must be the total number of outcomes in the sample space; $|E_1| + |E_2| + |E_3| = 2^{2n}$. By symmetry we see that $|E_1| = |E_3|$ and so $2|E_1| + |E_2| = 2^{2n}$.

   Now consider A's final throw. This brings the total number of throws to $2n+1$ and hence the size of the sample space to 2^{2n+1}. If event E_1 has occurred then A has won with either a head or a tail on the final throw; this gives $2|E_1|$ outcomes. If E_2 has occurs then A needs to throw a head to win; giving $|E_2|$ outcomes. Thus the total number of ways that A can win is $2|E_1| + |E_2|$. We established above that $2|E_1| + |E_2| = 2^{2n}$. Thus $P(A\text{wins}) = 2^{2n}/2^{2n+1} = 1/2$.

   Another way is to use a few probability generating functions. Suppose we have m coin tosses for A and n for B. It is obvious by inspection that for $r = -n, -n+1, \ldots, m$ the coefficient of t^r in

$$(1+t)^m(1+1/t)^n/2^{m+n} = (1+t)^{m+n}/[2^{m+n}t^n]$$

is the probability that there are exactly r more heads for A than for B. Expanding the binomial on the right we evaluate the probability as

$$\binom{m+n}{n+r}/2^{m+n}.$$

This is the probability of $n+r$ heads in $m+n$ tosses. So the probability that A has more heads than B is the same as the probability of more than n heads in $m+n$ tosses of the coin. This is obviously 0.5 if $m = n+1$.

It is easy to obtain directly the probability given above of exactly r more heads for A than for B. The probability of r more heads for A than for B is the sum for $s = \max(0, -r), 1, \ldots, \min(n, m-r)$ of the probabilities that we have $s+r$ heads in m tosses and also s heads from n tosses. The probabilities summed are the same as the probabilities that there are $s+r$ heads in m tosses and also $n-s$ heads from n tosses. The sum of those probabilities is the same as the probability of $n+r$ heads in $m+n$ tosses.

5. It is easiest to use a sample space which has equally likely outcomes, and then to count outcomes to find the required probabilities. Let's number the positions in the row from 1 to 6. A and B are equally likely to occupy any 2 positions that are distinct. We can use a diagram to show the outcome space - each circle shows a possible position for A and B, and all 30 outcomes are equally likely.

	Position of A					
	1	2	3	4	5	6
Position of B						
1		○	○	○	○	○
2	○		○	○	○	○
3	○	○		○	○	○
4	○	○	○		○	○
5	○	○	○	○		○
6	○	○	○	○	○	

Now we can mark each outcome with the number of people between A and B. This gives

	Position of A					
	1	2	3	4	5	6
Position of B						
1		0	1	2	3	4
2	0		0	1	2	3
3	1	0		0	1	2
4	2	1	0		0	1
5	3	2	1	0		0
6	4	3	2	1	0	

Then counting the number of outcomes for each case gives

$$P(0) = 10/30 = 1/3$$
$$P(1) = 8/30 = 4/15$$
$$P(2) = 6/30 = 1/5$$
$$P(3) = 4/30 = 2/15$$
$$P(4) = 2/30 = 1/15.$$

Now it is easy to generalise to the case of n positions. There are $n^2 - n = n(n-1)$ equally likely positions for A and B, and $2(n-r-1)$ of these have r people between A and B. So

$$P(r \text{ people between } A \text{ and } B) = \frac{2(n-r-1)}{n(n-1)}$$

for $r = 0, 1, 2, \ldots, n-2$.

If the positions for A and B are in a ring, and we look clockwise for the number of people between them, the same diagram will work, but we need to put in different labels for the numbers between.

	Position of A					
	1	2	3	4	5	6
Position of B						
1		4	3	2	1	0
2	0		4	3	2	1
3	1	0		4	3	2
4	2	1	0		4	3
5	3	2	1	0		4
6	4	3	2	1	0	

In this case it is obvious that all numbers of people between 0 and $n-2$ have the same probability. This is also true in the general case of n positions. It is often true that arrangements on a circle have simpler properties than arrangements on a line.

6. The urn contains 8 distinct disks.

	A	B	C	D
red	$\circ$	$\circ$		
green	$\circ$	$\circ$		
blue	$\circ$	$\circ$	$\circ$	$\circ$

By inspection we can see;

(a) $P(A) = 3/8$, $P(\text{red}) = 1/4$ and $P(A \cap \text{red}) = 1/8$ hence not independent,

(b) $P(A \cup D) = 1/2$, $P(\text{red}) = 1/4$ and $P((A \cup D) \cap \text{red}) = 1/8$ hence independent,

(c) $P(A \cup \text{blue}) = 3/4$, $P(\text{red}) = 1/4$ and $P((A \cup \text{blue}) \cap \text{red}) = 1/8$ hence not independent,

7. (a) We can use a sample space of eight equally likely outcomes if we take into account birth order. Using M and F for boys and girls, we can write

$$\Omega = \{MMM, MMF, MFM, FMM, MFF, FMF, MFF, FFF\}$$

Then $A = \{MMF, MFM, FMM, MFF, FMF, MFF\}$,
$B = \{MMM, MMF, MFM, FMM\}$ and $A \cap B = \{MMF, MFM, FMM\}$. Counting the outcomes in the events gives

$$P(A) = 6/8 = 3/4$$
$$P(B) = 4/8 = 1/2$$
$$P(A \cap B) = 3/8$$

Obviously $P(A \cap B) = P(A)P(B)$, so $A \perp B$. It is hard to see this independence intuitively. One needs to verify it to be sure.

(b) For four children families there are 16 equally likely family outcomes. Just 2 of these have all the children of the same gender, so $P(A) = 14/16 = 7/8$, and there is $\binom{4}{0} = 1$ family with all boys and $\binom{4}{1} = 4$ families with 1 girl. All the other families have more than 1 girl, so $P(B) = 5/16$. There are 4 families with children of both genders and no more than 1 girl, so $P(AB) = 4/16 = 1/4$. In this case there is not independence between A and B because

$$P(A \cap B) = \frac{1}{4} \neq \frac{7}{8} \times \frac{5}{16} = P(A)P(B).$$

8. (a) Since $A \cap B$ and $A \cap B^c$ are disjoint,

$$A = (A \cap B) \cup (A \cap B^c)$$

implies

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

or

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

Since $A \perp B$, $P(A \cap B) = P(A)P(B)$, and so the last equation implies

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] \\ &= P(A)P(B^c). \end{aligned}$$

So $A \perp B^c$.

(b) If

$$P(A|B) < P(A)$$

then since $P(A|B) = P(A \cap B)/P(B)$, it follows that

$$\frac{P(A \cap B)}{P(B)} < P(A)$$

and so that

$$P(A \cap B) < P(A)P(B).$$

Since $P(A) > 0$ we can divide both sides by $P(A)$ to get

$$\frac{P(A \cap B)}{P(A)} < P(B)$$

which is

$$P(B|A) < P(B).$$

Notice that there is an intuitive interpretation of this result. It says that if A makes B less probable, then B makes A less probable.

(c) This is a false statement. It says that A and B are conditionally independent given event C. To show that this does not imply in general that A and B are independent, you must find an explicit counter-example. One such is found by choosing $A = B = C$ and $P(A) < 1$. Then $P(A|C) = P(B|C) = P(A \cap B)|C) = 1$, and so the conditional independence trivially holds true. However, $P(A \cap B) = P(A)$ and $P(A)P(B) = P(A)^2$, so there is not independence between A and B, for that implies $P(A) = P(A)^2$, which is false as $P(A) \neq 0$ or 1.

(d) The statement in the question says that B has the same probability if A occurs as if A does not occur. One would expect that this should imply A and B independent, and it does!

$$P(B|A) = P(B|A^c)$$

implies

$$\frac{P(A \cap B)}{P(A)} = \frac{P(A^c \cap B)}{P(A^c)}$$

Adding the numerators and denominators, this implies

$$\frac{P(A \cap B)}{P(A)} = \frac{P(A \cap B) + P(A^c \cap B)}{P(A) + P(A^c)}$$

which is

$$\frac{P(A \cap B)}{P(A)} = \frac{P(B)}{1}$$

Cross-multiplying gives

$$P(A \cap B) = P(A)P(B)$$

which shows that $A \perp B$.

9. Let A_i be the event that ith face does not appear. We will use the formulae for unions

$$P(\bigcup_{i=1}^{6} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots - P(A_1 \cap \ldots \cap A_6). \quad (1)$$

If we throw the die n times then

$P(A_1) = (5/6)^n$ $\quad$ $P(A_1 \cap A_2) = (4/6)^n$ $\quad$ $P(A_1 \cap A_2 \cap A_3) = (3/6)^n$

$P(A_1 \cap \ldots \cap A_4) = (2/6)^n$ $\quad$ $P(A_1 \cap \ldots \cap A_5) = (1/6)^n$ $\quad$ $P(A_1 \cap \ldots \cap A_6) = 0$

Let E be event each face appears at least once. Using (1) and exploiting symmetry yields;

$$\begin{aligned} P(E) &= 1 - P(A_1 \cup \ldots \cup A_6) \\ &= 1 - 6(5/6)^n + 15(2/3)^n - 20(1/2)^n + 15(1/3)^n - 6(1/6)^n. \end{aligned}$$

10. For X and Y mutually exclusive $P(X \cup Y) = P(X) + P(Y)$. For general events A and B;

(a) $A \cup A^c = \Omega \Rightarrow P(A \cup A^c) = 1$.
A and A^c mutually exclusive $\Rightarrow P(A \cup A^c) = P(A) + P(A^c)$.
Thus $P(A) + P(A^c) = 1$ and so $P(A^c) = 1 - P(A)$.

(b) $B = (A \cap B) \cup (A^c \cap B) \Rightarrow P(B) = P((A \cap B) \cup (A^c \cap B))$
$A \cap B$ and $A^c \cap B$ mutually exclusive $\Rightarrow P((A \cap B) \cup (A^c \cap B)) = P(A \cap B) + P(A^c \cap B)$
Thus $P(A \cap B) + P(A^c \cap B) = P(B)$ and so $P(A^c \cap B) = P(B) - P(A \cap B)$.

(c) $A \cup B = A \cup (A^c \cap B) \Rightarrow P(A \cup B) = P(A \cup (A^c \cap B))$.
A and $(A^c \cap B)$ mutually exclusive $\Rightarrow P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$.
Thus $P(A \cup B) = P(A) + P(A^c \cap B) = P(A) + P(B) - P(A \cap B)$ by part 10b.

(d) $(A \cup B) \cap (A^c \cup B^c) = (A \cap B^c) \cup (A^c \cap B) \Rightarrow P((A \cup B) \cap (A^c \cup B^c)) = P((A \cap B^c) \cup (A^c \cap B))$.
$(A \cap B^c)$ and $(A^c \cap B)$ are mutually exclusive so $P((A \cap B^c) \cup (A^c \cap B)) = P(A \cap B^c) + P(A^c \cap B)$.
Thus $P((A \cup B) \cap (A^c \cup B^c)) = P(A \cap B^c) + P(A^c \cap B) = P(A) + P(B) - 2P(A \cap B)$ by 10b.

We could interpret the event $(A \cup B) \cap (A^c \cup B^c)$ as the event that either A or B happens but not both.

11. (a) Partition B_1 vicious $P(B_1) = 0.1$
B_2 not vicious $P(B_2) = 0.9$

If R is the event that a randomly chosen worm is red, we can see from the questions that $P(R|B_1) = 4/5$ and $P(R|B_2) = 1/4$.

Law of total probability to get probability worm is red;

$$\begin{aligned} P(R) &= P(R|B_1)P(B_1) + P(R|B_2)P(B_2) \\ &= 4/5 \times 0.1 + 1/4 \times 0.9 = 0.305. \end{aligned}$$

Bayes' theorem to get probability that a worm is vicious given that it is red;

$$\begin{aligned} P(B_1|R) &= P(R|B_1)P(B)/P(R) \\ &= 4/50 \times 200/61 = 16/61 = 0.262. \end{aligned}$$

(b) Define events R_1 - worm red; R_2 - parent red; R_3 - grandparent red. From part 11a $P(R_1) = 0.305 = P(R_2) = P(R_3)$. From question $P(R_1|R_2) = 3/4$, $P(R_1|R_3) = 2/3$ and $P(R_1|R_2 \cap R_3) = 4/5$.

i. Probability red and parent red is given by
$P(R_1 \cap R_2) = P(R_1|R_2)P(R_2) = 3/4 \times 0.305 = 0.22875$.

ii. Probability red or parent red or grandparent red given by
$P(R_1 \cup R_2 \cup R_3) = P(R_1) + P(R_2) + P(R_3) - P(R_1 \cap R_2) - P(R_2 \cap R_3) - P(R_1 \cap R_3) + P(R_1 \cap R_2 \cap R_3)$.
Elements of this equation:
$P(R_1 \cap R_2) = 0.22875$
$P(R_2 \cap R_3) = 0.22875$ (parent to grandparent $\sim$ child to parent)
$P(R_1 \cap R_3) = P(R_1|R_3)P(R_3) = 0.305 \times 2/3 = 0.20333$
$P(R_1 \cap R_2 \cap R_3) = P(R_1|R_2 \cap R_3)P(R_2 \cap R_3) = 4/5 \times 0.22875 = 0.183$
$\Rightarrow P(R_1 \cup R_2 \cup R_3) = 0.4371$.

12. This question just uses the law of total probability.

(a) When we transfer just one disk;

Partition: R_1 transfer red disk, $P(R_1) = 4/9$
G_1 transfer green disk, $P(G_1) = 5/9$

Let R_2 be the event that the disk we draw from the second urn is red then

$$\begin{aligned} P(R_2) &= P(R_2|R_1)P(R_1) + P(R_2|G_1)P(G_1) \\ &= 6/10 \times 4/9 + 5/10 \times 5/9 = 49/90 = 0.544. \end{aligned}$$

(b) Transferring two disks;

Partition: RR transfer 2 red disks, $P(RR) = 4/9 \times 3/8 = 1/6$
RG transfer 1 red and 1 green disk, $P(RG) = 2 \times 4/9 \times 5/8 = 5/9$
GG transfer 2 green disks, $P(GG) = 5/9 \times 4/8 = 5/18$

$$\begin{aligned} P(R_2) &= P(R_2|RR)P(RR) + P(R_2|RG)P(RG) + P(R_2|GG)P(GG) \\ &= 7/11 \times 1/6 + 6/11 \times 5/9 + 5/11 \times 5/18 = 53/99 = 0.535 \end{aligned}$$

